Algebraic Geometry Lecture 19 – Affine Recappage

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What is Algebraic Geometry?

It is the study of...

- affine varieties
- projective varieties
- quasi-projective varieties
- varieties
- schemes.

Affine Varieties

Let k be an algebraically closed field (we'll always assume this unless explicitly stated otherwise). Affine n-space is

$$\mathbb{A}^n = \{(a_1, \ldots, a_n) \in k^n\}.$$

We want to define affine varieties.

Anecdote: Andrew's commutative algebra lecturer once said

$$\text{Algebra} \quad \Leftrightarrow \quad \text{Geometry}.$$

1) Algebra $\Rightarrow$ Geometry.

Let $f_1, \ldots, f_m \in k[x] = k[x_1, \ldots, x_n]$. An affine algebraic set V associated with these polynomials is

$$V(f_1, \ldots, f_m) = \{x \in \mathbb{A}^n \mid f_1(x) = f_2(x) = \ldots = f_m(x) = 0\}.$$

We could've started with the ideal $(f_1, \ldots, f_m) \subset k[x]$. In fact we could start with any ideal $I \subset k[x]$ and construct $V(I)$. This is true by Hilbert's basis theorem, which says any ideal $I \subset k[x]$ is finitely generated.

2) Geometry $\Rightarrow$ Algebra.

Suppose V is an algebraic set – a geometric object in n-space. Define the ideal of V to be

$$I(V) = \{f \in k[x] \mid f(x) = 0 \text{ for all } x \in V\}.$$

What's the correspondence betwixt V and I? Certainly $V(I(U)) = U$, however in general $I(V(J)) \neq J$. We get a "nice" correspondence when we look at affine varieties.

Suppose V is an algebraic set and $V = W_1 \cup W_2$ where W_1, W_2 are proper algebraic subsets of V. Then V is called reducible. If V isn't reducible then it's called irreducible. Equivalently, V is irreducible if $I(V)$ is a prime ideal. We'll call an irreducible algebraic set an (affine algebraic) variety.

E.g. Consider $V : x^2 = 0$ in $\mathbb{A}^1$. So $V = \{0\}$, which is irreducible and hence an affine variety. And $I(V) = (x)$ which is a prime ideal. But if $J = (x^2)$ then $I(V(J)) = (x) \neq J$.

Fact: (Nullstellensatz) $I(V(J)) = \mathrm{rad}(J) = \{f \in k[x] \mid f^r \in J \text{ for some } r \in \mathbb{N}\}$.

Fact: Prime ideals are radical, i.e. if P is a prime ideal then $P = \mathrm{rad}(P)$.

Moral: When we use varieties (i.e. prime ideals) we get a nice correspondence, whence: Algebra iff Geometry.

Functions

Want to know what kind of interesting functions $f : V \to k$ we can get.

Geometric approach. A function $f : V \to k$ is regular if there exists a polynomial $F(x) \in k[x]$ such that $f(x) = F(x)$ for all $x \in V$. Note that F is not unique. Let $\mathcal{O}(V)$ denote the ring of regular functions.

Algebraic approach. The affine coordinate ring is the integral domain

$$k[V] := k[x]/I(V).$$

Fact: $\mathcal{O}(V) \cong k[V]$.

Because $k[V]$ is an integral domain we can define its field of fractions to be $k(V)$, the function field. Elements of $k(V)$ are called rational functions and have the form $\varphi = f/g$ for $f, g \in k[V]$. The dimension of V is then defined to be the transcendence degree of $k(V)$ over k – i.e. the size of the largest algebraically independent subset over k.